J. Fluid Mech. (1981), *vol.* 110, *pp.* 249–253
Printed in Great Britain

The focusing of weak shock waves at an axisymmetric arête

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(Received 20 November 1980)

The focusing of a weak and nearly plane shock wave at an axisymmetric arête is examined. The initial-value problem governing the flow in the focal region is derived. A similitude is introduced which shows that the details of the flow depend on the initial shape and strength of the shock only through a simple rescaling of the variables. The dependence of the pressure coefficient on the physical parameters is given. It is seen that the maximum pressure coefficient is proportional to a parameter measuring the rate of focusing of the shock and to the square root of a parameter measuring the initial strength of the shock.

1. Introduction

In recent years there has been considerable interest in focusing shock waves. In his theory of shock dynamics, Whitham (1957, 1959) has described the behaviour of shock waves of moderate strength. An important experimental study of these phenomena is due to Sturtevant & Kulkarny (1976). In this study the focusing of a wide variety of shock waves at point foci, caustic surfaces and arêtes was delineated. The focusing of weak shock waves at smooth caustics has been described analytically by Guiraud (1965), Hayes (1968) and Pechuzal & Kevorkian (1977). Cramer & Seebass (1978) have examined the focusing of weak shock waves at a two-dimensional arête. Cramer (1980) has shown that this two-dimensional theory can also describe the focusing process at a three-dimensional arête, provided the two principal radii of curvature of the shock surface are not identical.

In this paper, we describe the focusing of a weak shock at an axisymmetric arête. Because of the three-dimensional symmetry of the problem, the resultant amplifications are expected to be much greater than those encountered at a two or three-dimensional arête. The only exceptions to this are the point focus and certain singular cases of the present theory. The general procedure of Cramer & Seebass (1978) is used to derive the initial-value problem governing the flow in the vicinity of the arête. A similitude is obtained which shows that the details of the focusing problem are independent of the physical parameters of the problem; this also allows us to relate the pressure coefficient directly to the initial strength and geometry of the shock. In order to save space, the next section simply outlines the derivation of the key results and emphasizes the differences between these and the two-dimensional theory.

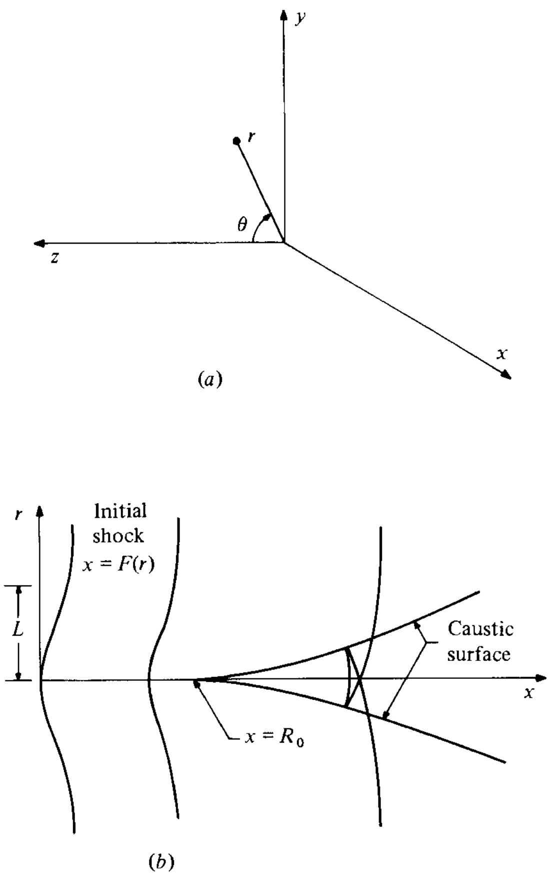

FIGURE 1. (a) Co-ordinate system. $z = r\cos\theta$, $y = r\sin\theta$. (b) Behaviour of a typical shock surface in a θ = constant plane. The second caustic surface degenerates to the line $r = 0$, $x \geqslant R_0$.

2. Outline of solution

It will be useful to refer to the Cartesian and cylindrical co-ordinate systems depicted in figure 1 (a). The direction of propagation is parallel to the positive x axis. At $t = 0$, the equation of the shock surface is given by

$$x = f(y,z) = F(r),$$

where $r \equiv (y^2+z^2)^{\frac{1}{2}}$; the focusing of a typical shock surface is indicated in figure 1 (b). As in the two-dimensional theory, it will be assumed that the shock surface is approximately plane. The quantity $\delta \equiv L/R_0$ will therefore be taken to be small. Here L is defined by $F''(L) \equiv 0$ and $R_0 \equiv [F''(0)]^{-1}$ is the value of either of the principal radii of curvature at $y = 0$ and $z = 0$. In the usual way, $F''(r) \equiv d^2F(r)/dr^2$. We will also need to require that

$$\kappa \equiv -R_0 L^2 F^{\text{iv}}(0) > 0$$

be finite and non-zero; here $F^{\text{iv}}(r) \equiv d^4F/dr^4$. The strength of the shock will be taken to be small; we will therefore require that

$$\epsilon \equiv c_p(0^-, 0, 0, 0)$$

be small. Here $c_p \equiv c_p(x,y,z,t)$ is the pressure coefficient. For the pressure levels and gradients encountered here, the flow may be regarded as irrotational; the equations of motion may therefore be recast in terms of a velocity potential, $\phi = \phi(x,y,z,t)$. The fluid will be taken to be an ideal gas; the equations of motion may therefore be written

$$\phi_{tt} + 2\phi_x\phi_{xt} + 2\phi_y\phi_{yt} + 2\phi_z\phi_{zt} + 2\phi_x\phi_y\phi_{xy} + 2\phi_x\phi_z\phi_{xz} + 2\phi_y\phi_z\phi_{yz}$$
$$= (a^2 - \phi_x^2)\,\phi_{xx} + (a^2 - \phi_y^2)\,\phi_{yy} + (a^2 - \phi_{zz}^2)\,\phi_{zz}, \quad (1)$$

where the sound speed, a, is given by the Bernoulli equation

$$a^2 = a_0^2 - (\gamma - 1)\,[\phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2)],$$

where a_0 is the sound speed of the undisturbed medium and γ is the ratio of specific heats. The initial conditions for (1) will be

$$\phi(x,y,z,0) = \phi_0(x,y,z), \quad \phi_t(x,y,z,0) = \phi_1(x,y,z). \quad (2)$$

The gas ahead of the shock is assumed to be uniform and at rest. Therefore, ϕ_0 and ϕ_1 are taken to be zero ahead of the shock, i.e. for $x > f(y,z) \equiv F(r)$. Behind the shock, they must be consistent with the usual shock jump conditions and the assumption of symmetry about the x axis.

For times $t = O(L/a_0)$ and $\epsilon = o(\delta^2)$, it may be shown that (1) may be linearized to yield the three-dimensional wave equation. As in the two-dimensional theory, the solution which satisfies the initial conditions (2) is given by the Poisson integral formula. The linear theory may be used to calculate the caustic surfaces associated with the initial shock surface. It is well known that there are generally two distinct caustic surfaces corresponding to any smooth surface, see e.g. Friedlander (1958). It is interesting to note that, in this axisymmetric problem, one of the caustic sheets degenerates into a line along the x axis, i.e. $y = 0$, $z = 0$, $x \geqslant R_0$. The other caustic sheet is simply the axisymmetric version of the caustic surface described by Cramer & Seebass (1978). It may be shown that the first caustic, i.e. the line caustic, corresponds to the intersection of rays originating at the same value of r on the initial shock surface but different values of θ and the second sheet corresponds to the intersection of rays originating in a constant θ plane but at different values of r.

The linear theory, of course, fails at the caustic surfaces. When the shock first touches the caustic, i.e. when $t = R_0/a_0$, the pressure coefficient is proportional to $|R_0 - x|^{-\frac{1}{2}}$ along the axis of symmetry and $r^{-\frac{2}{3}}$ in the $x = R_0$ plane. These singularities are seen to be much stronger than those corresponding to the two-dimensional arête or smooth caustic. At the arête, as well as the smooth portions of the caustic, nonlinear effects predominate and we need to find a second approximation to (1) which is valid in the vicinity of the arête, i.e. $x \approx R_0$, $y \approx 0$, $z \approx 0$. The initial condition or incoming signal is obtained by matching this inner solution to the linear or outer solution. When the procedure of Cramer & Seebass (1978) is applied to the present problem, we find that the flow near the arête is governed by the initial-value problem

$$2\hat{\phi}_{X\hat{t}} + (\gamma+1)\,\hat{\phi}_X\hat{\phi}_{XX} + \hat{\phi}_{\hat{y}\hat{y}} + \hat{\phi}_{\hat{z}\hat{z}} = 0, \quad (3)$$

where

$$\hat{\phi} \sim -\frac{\hat{t}}{\kappa}G(\sigma,\Gamma) \quad \text{as} \quad \hat{t} \to -\infty.$$

Here

$$G(\sigma,\Gamma) = -\frac{3}{\pi}\int_{q_l}^{q_u}[-(q^2+1) + (1+\sigma+\Gamma q)^{\frac{1}{2}}]^{\frac{1}{2}}\,dq,$$

and

$$\sigma \equiv -\tfrac{2}{3}\kappa\frac{\chi}{(-\hat{t})^2}, \quad \Gamma \equiv 24\kappa^{\frac{1}{2}}\frac{\hat{r}}{(-6\hat{t})^{\frac{3}{2}}},$$

where $\hat{r} \equiv (\hat{y}^2+\hat{z}^2)^{\frac{1}{2}}$. The limits q_u and q_l are the two real roots of

$$-(q^2+1)+(1+\sigma+\Gamma q)^{\frac{1}{2}} = 0$$

or, equivalently,

$$q^4+2q^2-\Gamma q-\sigma = 0.$$

As in the two- and three-dimensional cases, the incoming solution is self-similar in time, although of a different form. As one would expect, the incoming solution depends on $\hat{z}$ and $\hat{y}$ only through $\hat{r}$; we will therefore assume $\hat{\phi} = \hat{\phi}(\chi, \hat{t}, \hat{r})$ only and replace the $\hat{\phi}_{\hat{y}\hat{y}}+\hat{\phi}_{\hat{z}\hat{z}}$ term appearing in (3) by $\hat{\phi}_{\hat{r}\hat{r}}+\hat{r}^{-1}\hat{\phi}_{\hat{r}}$. The dimensional quantities are related to the scaled variables $\hat{\phi}, \chi, \hat{r}, \hat{t}$ by

$$\phi = \Delta^3\delta^3 La_0\hat{\phi}, \quad r = \Delta^{\frac{3}{2}}L\hat{r}, \quad t = \frac{R_0}{a_0}(1+\Delta\hat{t}), \quad x-a_0t = \Delta^2\delta L\chi,$$

where $\Delta \equiv (\epsilon/\delta^2)^{\frac{1}{2}} = o(1)$.

We now obtain the similitude for this problem by defining new variables ρ, τ, ρ and Φ as follows:

$$\chi = (\gamma+1)\,\xi, \quad \hat{t} = (\gamma+1)^{\frac{1}{2}}\kappa^{\frac{1}{2}}\tau, \quad \hat{r} = (\gamma+1)^{\frac{3}{4}}\kappa^{\frac{1}{4}}\rho, \quad \hat{\phi} = \left(\frac{\gamma+1}{\kappa}\right)^{\frac{1}{2}}\Phi.$$

In terms of these variables, the initial-value problem becomes

$$2\Phi_{\xi\tau}+\Phi_{\xi}\Phi_{\xi\xi}+\Phi_{\rho\rho}+\frac{1}{\rho}\Phi_{\rho} = 0 \tag{4}$$

with

$$\Phi \sim -\tau G(\sigma, \Gamma) \quad \text{as} \quad \tau \to -\infty.$$

In terms of these variables

$$\sigma = -\frac{2}{3}\frac{\xi}{\tau^2} \quad \text{and} \quad \Gamma = 24\frac{\rho}{(-6\tau)^{\frac{3}{2}}};$$

thus, there is no similarity parameter and $\Phi = \Phi(\xi, \tau, \rho)$ only. Except for a rescaling of the dependent and independent variables, the details of the focusing process are the same for all axisymmetric shock surfaces. An analogous similitude has been given by Cramer (1980) for two- and three-dimensional problems. The pressure coefficient $c_p \approx 2\phi_X/a_0$, where $X \equiv x-a_0t$, may now be written

$$c_p = 2\frac{\epsilon^{\frac{1}{2}}\delta}{[(\gamma+1)\kappa]^{\frac{1}{2}}}\phi_{\xi}(\xi, \tau, \rho).$$

Thus, the dependence of the pressure coefficient on the physical parameters is given explicitly.

The maximum pressure coefficient is therefore seen to be proportional to $\epsilon^{\frac{1}{2}}\delta$. If we compare this to a two-dimensional shock having the same ϵ and δ we see that the pressure levels obtained here are a factor of $(\epsilon/\delta^2)^{\frac{1}{6}}$ larger than those in a locally two-dimensional problem. A similar analysis can be made of the size of the focal region, i.e. the region in which nonlinear effects predominate. It is easily seen that the axisymmetric focusing yields a much larger focal region than that of the two-dimensional theory; this is apparently due to the rapid amplification of the axisymmetric problem.

3. Conclusion

The procedure of Cramer & Seebass (1978) has been used to derive the initial-value problem and similitude governing the flow in the vicinity of an axisymmetric arête. The pressure coefficient was found to be proportional to $\epsilon^{\frac{1}{2}}\delta$, where ϵ measures the initial strength of the shock and δ measures the rate of focusing. The details of the flow are given by (4); these are essentially independent of the physical parameters of the problem. As in the two-and three-dimensional problems, these results are only valid if $\epsilon = o(\delta^2)$ and $\delta = o(1)$.

This research was supported by the National Science Foundation through grant ENG79–11888.

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